Name:......................................

Student Number

Teacher:..................................

Pymble Ladies' College

Mathematics Extension 1
HSC Trial Examination
Term 3 2023

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks 70

SECTION 1 – 10 marks (pages 1-4)

- Attempt Questions 1-10
- Allow about 15 minutes for this section
- Answer each question on the multiple-choice answer sheet provided in the answer booklet.

SECTION II – 60 marks (pages 5-10)

- Attempt Questions 11-14
- Allow about 1 hours and 45 minutes for this section
- Answer each question in the appropriate space in the Answer Booklet. Extra writing pages are included at the end of each question.

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SECTION I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10

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1. Given the points $A(-2,-5)$ and $B(0, 7)$, which of the following represents the displacement vector $\overrightarrow{BA}$?

(A) $-2\underset{\sim}{i}-12\underset{\sim}{j}$

(B) $2\underset{\sim}{i}+12\underset{\sim}{j}$

(C) $-2\underset{\sim}{i}+2\underset{\sim}{j}$

(D) $2\underset{\sim}{i}+2\underset{\sim}{j}$

2. What is the anti-derivative of $\cos^2 2x$?

(A) $\dfrac{\cos^3 2x}{3}+C$

(B) $\dfrac{1}{2}\sin^2 2x+C$

(C) $\dfrac{x}{2}+\dfrac{\sin 4x}{8}+C$

(D) $\dfrac{1}{2}+\dfrac{1}{2}\cos 4x+C$

3. Which of the following expressions is the correct rate of change of the area of a circle (A) with respect to time (t)?

(A) $\dfrac{dr}{dt}=2\pi r\times\dfrac{dA}{dt}$

(B) $\dfrac{dr}{dt}=2\pi r\times\dfrac{dA}{dr}$

(C) $\dfrac{dA}{dr}=2\pi r\times\dfrac{dt}{dr}$

(D) $\dfrac{dA}{dt}=2\pi r\times\dfrac{dr}{dt}$

4. Which polynomial has a root at $x = 1$ of multiplicity 3?

(A) $x^3 - 3x + 2$

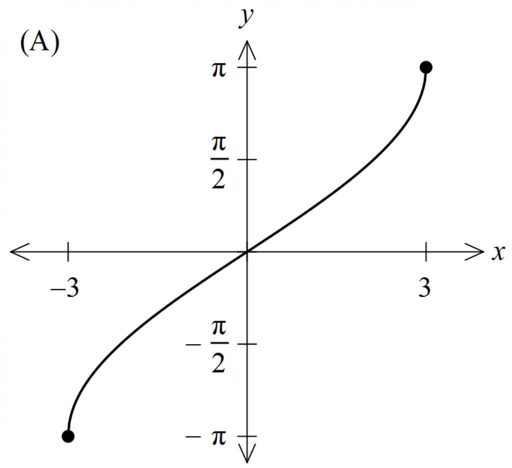
(B) $x^4 - 2x^3 + 2x - 1$

(C) $x^3 + 4x^2 + 5x + 2$

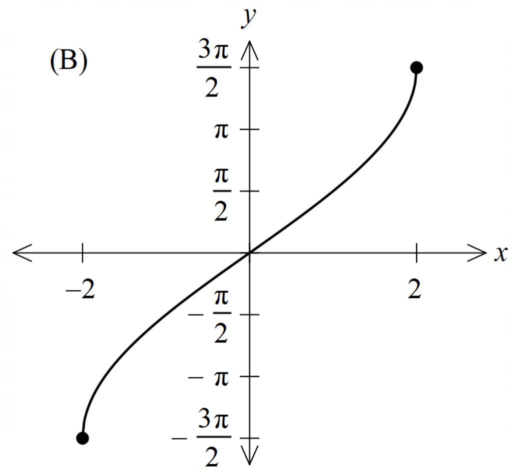
(D) $x^4 + 2x^3 - 2x - 1$

5. Which graph best represents $y = 3\sin^{-1}\dfrac{x}{2}$?

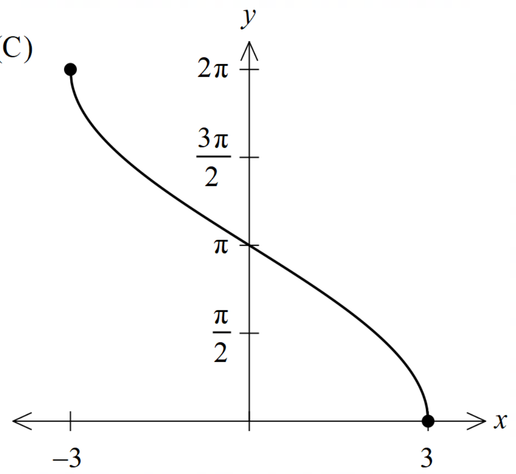
(A)



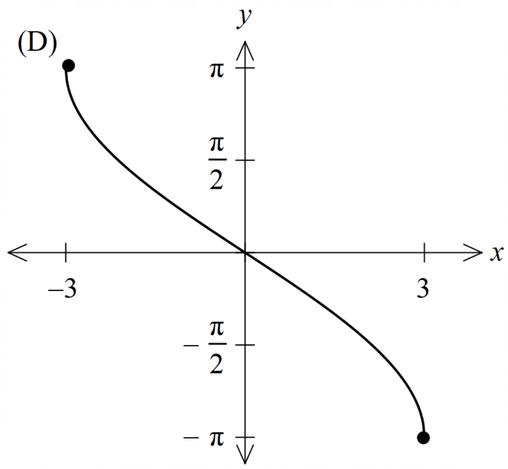
(B)



(C)



(D)



6. Which of the following is the derivative of $y = \tan^{-1}\left[2f(x)\right]$?

(A) $\dfrac{dy}{dx} = \dfrac{1}{1+\left[f(x)\right]^2}$

(B) $\dfrac{dy}{dx} = \dfrac{2}{1+4\left[f(x)\right]^2}$

(C) $\dfrac{dy}{dx} = \dfrac{f'(x)}{1+4\left[f(x)\right]^2}$

(D) $\dfrac{dy}{dx} = \dfrac{2f'(x)}{1+4\left[f(x)\right]^2}$

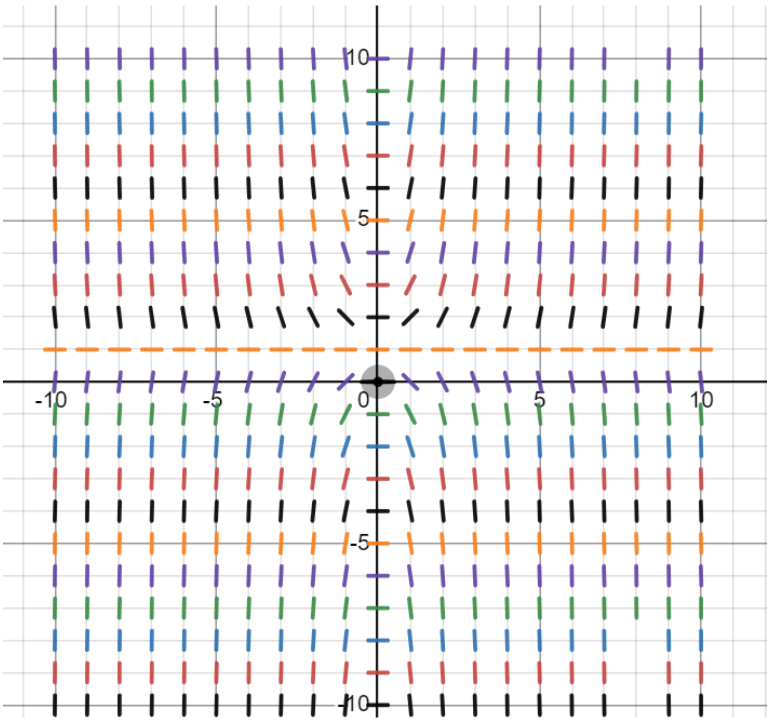
7. The direction field for a certain differential equation is shown below. Which of the following is the differential equation?

(A) $\dfrac{dy}{dx} = x^2 + y^2$

(B) $\dfrac{dy}{dx} = x^2 - y^2$

(C) $\dfrac{dy}{dx} = xy - x$

(D) $\dfrac{dy}{dx} = y^2 - x^2$



8. What is the value of $\sin^{-1}(\cos\alpha)$ where $\frac{3\pi}{2} < \alpha < 2\pi$?

(A) $2\pi - \alpha$

(B) $\alpha - 2\pi$

(C) $\frac{3\pi}{2} + \alpha$

(D) $\alpha - \frac{3\pi}{2}$

9. The equation $y = e^{ax}$ satisfies the differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0.$$

What are the possible values of a?

(A) $a = -2$ or $a = 3$

(B) $a = -1$ or $a = 6$

(C) $a = -3$ or $a = 2$

(D) $a = -6$ or $a = 1$

10. Students at a school are required to study 4 subjects from a selection of 10. The ten subjects are placed in two lines, each containing 5 subjects.
How many possible combinations of subjects exist if a student must take at least 1 subject from each line?

(A) 200

(B) 250

(C) 309

(D) 356

SECTION II

60 marks
Attempt Questions 11-14
Allow about 1 hour and 45 minutes for this section.

Answer these questions in the Answer Book provided.
Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (17 marks) **Marks**

(a) The polynomial $P(x) = 2x^4 - 11x^3 + 19x^2 - 13x + 3$ has roots α, β, γ and δ. **2**

Find the value of $\dfrac{1}{\alpha\beta\gamma} + \dfrac{1}{\alpha\beta\delta} + \dfrac{1}{\beta\gamma\delta} + \dfrac{1}{\alpha\gamma\delta}$

(b) Find $\displaystyle\int \frac{1}{4x^2 + 9}\, dx.$ **2**

(c) A square metal sheet is heated so that its side length is increasing at a uniform rate of 0.05 cms^{-1}. Find the rate its area is changing at the instant when the side length is 3.1 cm. **2**

(d) Find $\displaystyle\int_0^{\frac{\pi}{6}} \sin 4x \cos 2x\, dx.$ **3**

(e) Given $\underset{\sim}{a} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ and $\underset{\sim}{b} = \begin{bmatrix} -6 \\ 4 \end{bmatrix}$, find:

(i) $|\underset{\sim}{a}|$ **1**

(ii) $\underset{\sim}{a} \cdot \underset{\sim}{b}$ **1**

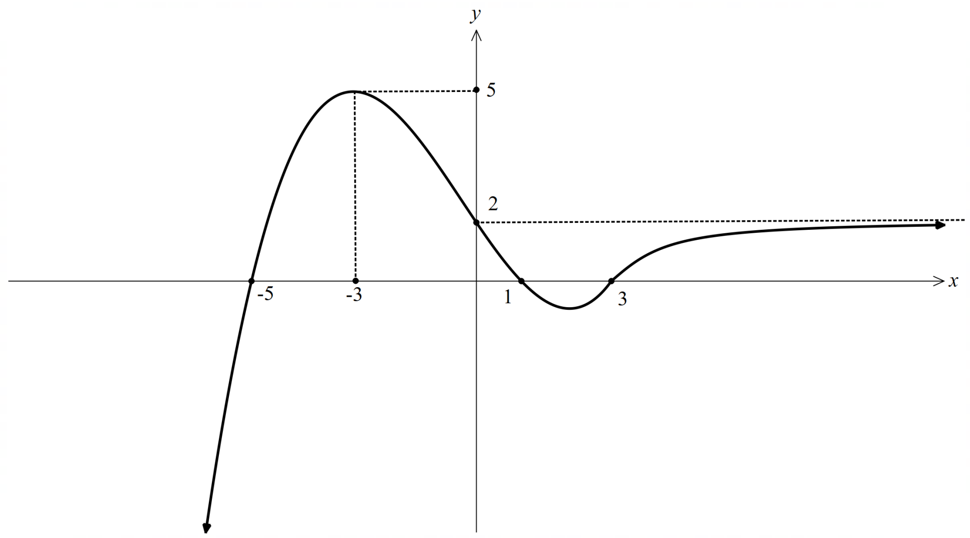
(iii) the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$, to the nearest degree. **1**

(iv) $\text{proj}_{\underset{\sim}{b}}\underset{\sim}{a}$ **2**

Question 11 continues on page 6

Question 11 (continued)

(f) The diagram shows the graph of $y = f(x)$ which has a horizontal asymptote at $y = 2$.



In your answer booklet draw clear sketches of the following.
Show all relevant information.

(i) $y = \left|f(x)\right|$ **1**

(ii) $y = \dfrac{1}{f(x)}$ **2**

End of Question 11

Question 12 (17 marks) **Marks**

(a) Find $\int \frac{3x}{\sqrt{9-x^2}}\,dx$ using the substitution $u = 9 - x^2$. **2**

(b) By expressing $\sqrt{3}\cos x - \sin x$ in the form $A\cos(x+\alpha)$, where $A > 0$ and $0 \le \alpha \le \frac{\pi}{2}$, solve $\sqrt{3}\cos x - \sin x + 1 = 0$, for $0 \le x \le 2\pi$. **3**

(c) Paddington Bear hits a golf ball with velocity V m/s at an angle of projection θ degrees to the horizontal.

The golf ball's position at any time t, is given by the position vector

$$\underset{\sim}{r}(t) = (Vt\cos\theta)\underset{\sim}{i} + \left(Vt\sin\theta - \frac{1}{2}gt^2\right)\underset{\sim}{j}. \quad \text{[Do not prove this]}$$

(i) Show that Paddington's golf ball reaches a maximum height of **2**

$$\frac{V^2\sin^2\theta}{2g} \text{ metres.}$$

(ii) What is the range of flight of the golf ball? **2**

(d) Prove by mathematical induction that $4^n + 14$ is divisible by 6 for all positive integers $n \ge 1$. **3**

(e) The polynomial $P(x) = x^3 + bx^2 + cx + d$ has roots 0, 4 and -4.

(i) Find b, c and d. **2**

(ii) Without calculus, sketch the graph of $y = P(x)$. **1**

(iii) Hence, or otherwise, solve the inequality $\frac{x^2 - 16}{x} \le 0$. **2**

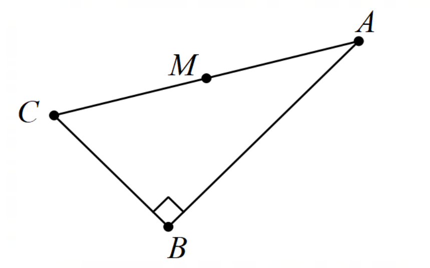
End of Question 12

Question 13 (14 marks) **Marks**

(a) What is the value of the term independent of x in the expansion of **2**

$$\left(2-\frac{3}{x^3}\right)\left(2x+\frac{5}{x^2}\right)^6 ?$$

(b) ABC is a right-angled triangle. M is the midpoint of the hypotenuse AC, as shown below. Let $\overrightarrow{AM} = \underset{\sim}{a}$ and $\overrightarrow{BM} = \underset{\sim}{b}$.



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(i) Express $\overrightarrow{AB}$ and $\overrightarrow{BC}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$. **2**

(ii) Prove that M is equidistant from the three vertices of ΔABC. **2**

(c) A relation is defined by the parametric equations

$$x = 1-\sin t,\ y = \cos t \sin 2t,\ \text{ for } 0 \le t \le \pi.$$

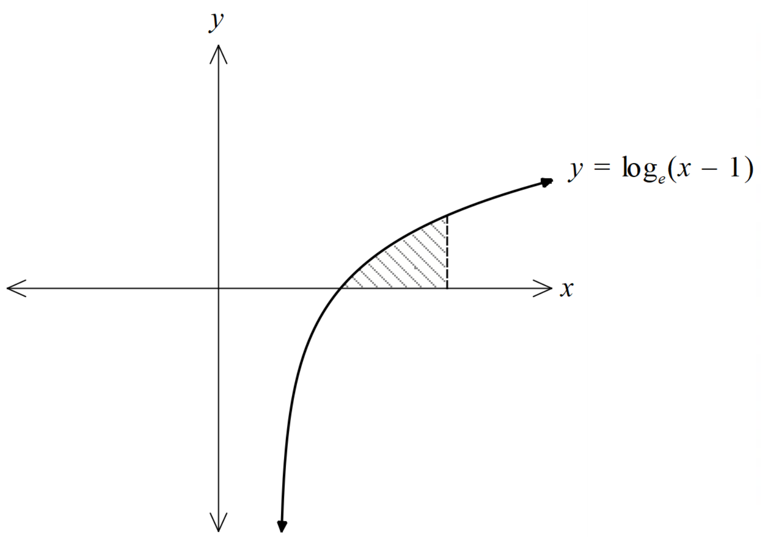
(i) Find the coordinates where the graph crosses the x and y axes. **2**

(ii) State the domain of the relation. **1**

(iii) Find the cartesian equation of the relation. **2**

Question 13 continues on page 9

(d) Part of the graph of $y = \log_e(x-1)$ is given.



(i) Find the value of y when $x = e + 1$. **1**

(ii) Find the shaded area, bounded by the curve, the x-axis and the line $x = e + 1$. **2**

End of Question 13

Question 14 (12 marks) **Marks**

(a) An area A cm^2 is occupied by a bacteria colony. The colony has a growth rate modelled by the logistic equation

$$\frac{dA}{dt} = \frac{1}{25}A(50 - A) \text{ where } t \geq 0 \text{ and is measured in days.}$$

At time $t = 0$, the area occupied by the bacteria colony is $\frac{1}{2}$ cm^2.

(i) Show that $\dfrac{1}{A(50-A)} = \dfrac{1}{50}\left(\dfrac{1}{A} + \dfrac{1}{50-A}\right)$. **1**

(ii) Solve the logistic equation and hence show that $A = \dfrac{50}{1+99e^{-2t}}$. **3**

(iii) What is the limiting area of the bacteria colony? **1**

(b) A Maths club has $2n$ members, half of which are male and half female. **2**
A Pi Day committee is made up of 3 members of this club. By considering all different combinations for this committee, show that

$$n\,{}^{n}C_2 + {}^{n}C_3 = \frac{1}{2}\,{}^{2n}C_3, \quad \text{where } n \geq 3.$$

(c) (i) Show that $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$. **2**

(ii) By using part (i) and letting $x = \cos\theta$ in the equation $16x^4 - 16x^2 + 2 - \sqrt{2} = 0$ **3**
show that $\cos^2\dfrac{\pi}{16}\cos^2\dfrac{7\pi}{16} = \dfrac{2-\sqrt{2}}{16}$.

END OF PAPER

NSW Education Standards Authority

2022 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}, \ r \neq 1$$

$$S = \frac{a}{1 - r}, \ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

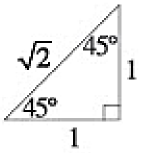
$$a^x = e^{x \ln a}$$

Trigonometric Functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

$$A = \frac{1}{2}ab\sin C$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

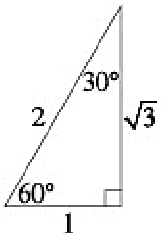


$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$



Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1+t^2}$

$$\cos A = \frac{1-t^2}{1+t^2}$$

$$\tan A = \frac{2t}{1-t^2}$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$$

$$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

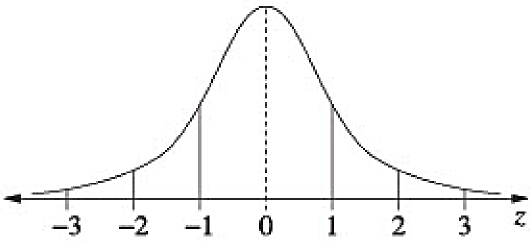
$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$$E(X) = \mu$$

$$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$$

Continuous random variables

$$P(X \le r) = \int_a^r f(x)\,dx$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Binomial distribution

$$P(X = r) = {}^nC_r p^r (1 - p)^{n-r}$$

$$X \sim \operatorname{Bin}(n, p)$$

$$\Rightarrow \ P(X = x)$$

$$= \binom{n}{x} p^x (1 - p)^{n-x}, \ x = 0, 1, \ldots, n$$

$$E(X) = np$$

$$\operatorname{Var}(X) = np(1 - p)$$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\frac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
$y = \frac{u}{v}$	$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\frac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\frac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\frac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\frac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\frac{dy}{dx} = (\ln a) f'(x) a^{f(x)}$
$y = \log_a f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{(\ln a) f(x)}$
$y = \sin^{-1} f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\frac{dy}{dx} = -\frac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{1+[f(x)]^2}$

Integral Calculus

$$\int f'(x)[f(x)]^n dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$$

$$\int f'(x)a^{f(x)}dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2 - [f(x)]^2}}dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2 + [f(x)]^2}dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f(x_1) + \cdots + f(x_{n-1})\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2 + y^2}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt + \alpha) + c$$

$$x = a\sin(nt + \alpha) + c$$

$$\ddot{x} = -n^2(x - c)$$

Name: Solutions -

Student Number:

Teacher's Name:

Pymble Ladies' College

Mathematics Extension 1
HSC Trial Examination
Term 3 2023

ANSWER BOOKLET

- **Put your name, Student Number and Teacher's name on this Booklet.**
- For Questions 1-10 use the Multiple-Choice Answer Sheet provided.
- Answer Questions 11-14 in the spaces allocated for each question as these spaces provide guidance for the expected length of your response.
- Write using a non-erasable black or blue pen. Black is preferred.
- Extra writing space is provided at the end of each question.
- Calculators approved by NESA may be used.

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Name:.....................................

Student Number:

Teacher's Name:..........................

Mathematics Extension 1
HSC Trial Examination

Section I – Multiple Choice Answer Sheet

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: $2 + 4 =$ (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ⊗ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word ***correct*** and drawing an arrow as follows.

A ⊗ B ⊗ ← *correct* C ○ D ○

1	A ●	B ○	C ○	D ○
2	A ○	B ○	C ●	D ○
3	A ○	B ○	C ○	D ●
4	A ○	B ●	C ○	D ○
5	A ○	B ●	C ○	D ○
6	A ○	B ○	C ○	D ●
7	A ○	B ○	C ●	D ○
8	A ○	B ○	C ○	D ●
9	A ○	B ○	C ●	D ○
10	A ●	B ○	C ○	D ○

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SECTION II 60 marks
For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.
Extra writing space is provided at the end of each question.

Question 11 (17 marks) **Marks**

(a) 2

$\frac{1}{\alpha\beta\gamma} + \frac{1}{\alpha\beta\delta} + \frac{1}{\beta\gamma\delta} + \frac{1}{\alpha\gamma\delta}$

$= \frac{\delta + \gamma + \alpha + \beta}{\alpha\beta\gamma\delta}$ $\quad \alpha + \beta + \gamma + \delta = 11/2$

$\alpha\beta\gamma\delta = 3/2$

$= \frac{11}{3}$

2	correct
1	One error in process

(b) 2

$\int \frac{1}{4x^2+9}\,dx$

$= \frac{1}{4}\int \frac{1}{x^2 + 9/4}\,dx$

$= \frac{1}{4} \times \frac{1}{3/2} \tan^{-1}\frac{x}{3/2} + C$

$= \frac{1}{6}\tan^{-1}\frac{2x}{3} + C$

2	correct
1	Constant correct or $\frac{2x}{3}$ correct

Question 11 continues on page 2

(c) 2

Let the side length be x

$\frac{dx}{dt} = 0.05$ cm/s $A = x^2$

$\frac{dA}{dx} = 2x$

2	correct
1	Expression for Area, derivative and related rate correct

$$\frac{dA}{dt} = \frac{dA}{dx} \times \frac{dx}{dt}$$

$$= 2 \times 3.1 \times 0.05$$

$$= 0.31 \text{ cm}^2/\text{s}$$

(d) 3

$$\int_0^{\pi/6} \sin 4x \cos 2x \, dx$$

$$= \frac{1}{2} \int_0^{\pi/6} \left(\sin(4x + 2x) + \sin(4x - 2x)\right) dx$$

$$= \frac{1}{2} \int_0^{\pi/6} (\sin 6x + \sin 2x) \, dx$$

$$= \frac{1}{2} \left[\frac{-\cos 6x}{6} - \frac{\cos 2x}{2} \right]_0^{\pi/6}$$

$$= \frac{1}{2} \left\{ \frac{-\cos \pi}{6} - \frac{\cos \pi/3}{2} - \left(\frac{-\cos 0}{6} - \frac{\cos 0}{2} \right) \right\}$$

$$= \frac{1}{2} \left\{ \frac{1}{6} - \frac{1}{4} + \frac{1}{6} + \frac{1}{2} \right\}$$

$$= \frac{7}{24}$$

3	Correct
2	Correct use of products to sums and integration error in calc
1	Correct sums to products.

Question 11 continues on page 3

Question 11 continued. **Marks**

(e)

(i) 1

$|\underset{\sim}{a}| = \sqrt{(-2)^2 + 3^2}$ 1 r/w

$= \sqrt{13}$

(ii) 1

$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos\theta = x_1 x_2 + y_1 y_2$ 1 r/w

$= -2 \times -6 + 3 \times 4$

$= 24$

(iii) 1

$\sqrt{13} \times \sqrt{52} \cos\theta = 24$ 1 r/w

$\cos\theta = 12/13$

$\theta = 22.61986495$

$\approx 23^\circ$ (nearest deg.)

(iv) 2

$\text{Proj}_{\underset{\sim}{b}} \underset{\sim}{a} = \dfrac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{b}|^2} \underset{\sim}{b}$

2	Correct
1	Error in calculation

$= \dfrac{24}{52}(-6\underset{\sim}{i} + 4\underset{\sim}{j})$

$= \dfrac{6}{13}(-6\underset{\sim}{i} + 4\underset{\sim}{j})$

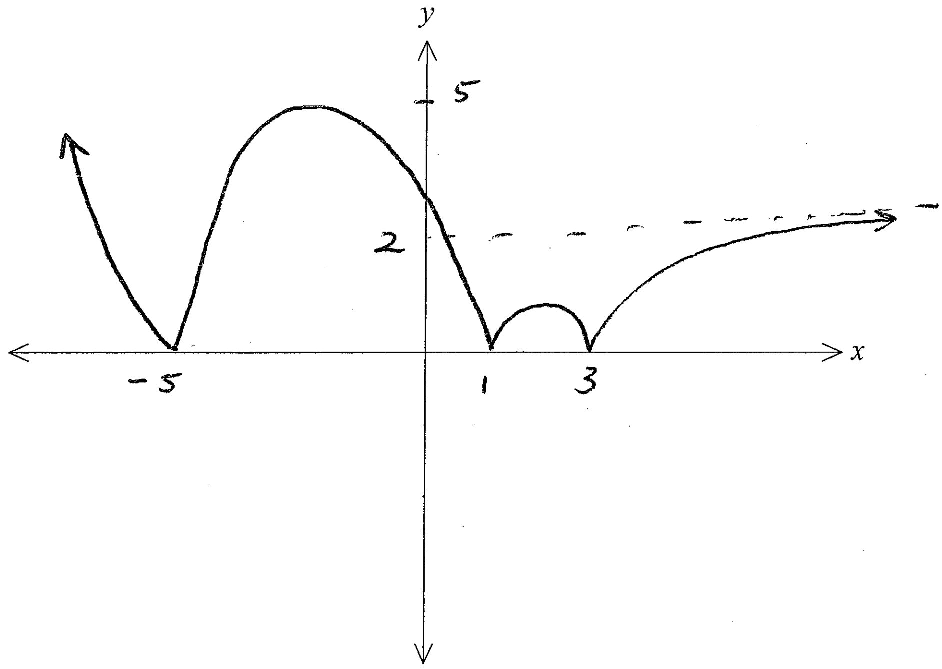
Question 11 continues on page 4

Question 11 continued **Marks**

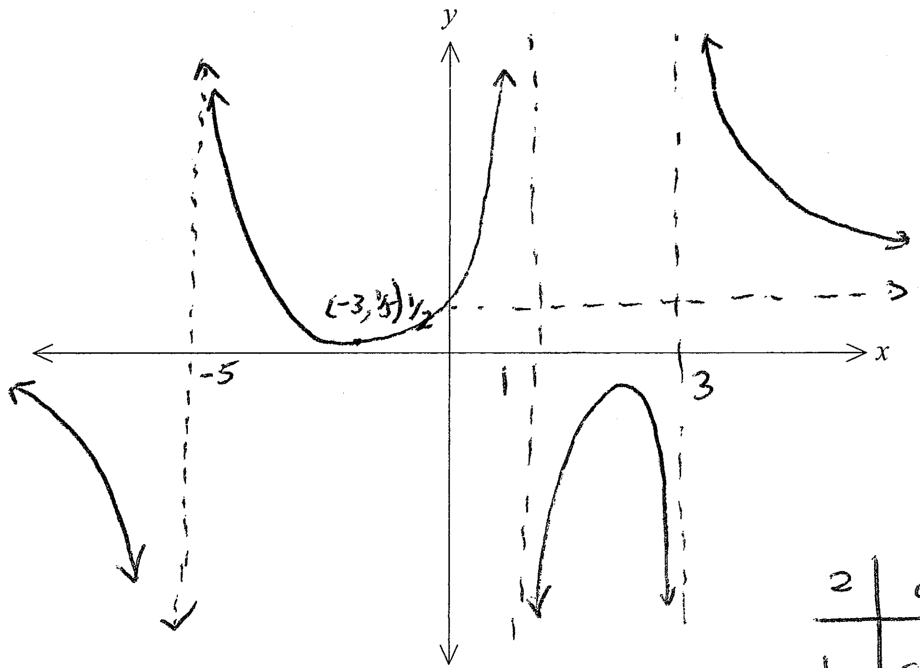
(f)

1 r/w

(i) **1**



(ii) **2**



2	Correct
1	Correct asymptotes and some pieces correct

End of Question 11

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Question 12 (17 marks) **Marks**

(a) 2

$$\int \frac{3x}{\sqrt{9-x^2}}\,dx \qquad u = 9 - x^2$$

$$du = -2x\,dx$$

$$* = \frac{3}{-2}\int \frac{du}{\sqrt{u}}$$

$$= -\tfrac{3}{2}\left[2u^{1/2}\right] + C$$

$$= -3\sqrt{u} + C$$

$$= -3\sqrt{9-x^2} + C$$

2	Correct
1	Correct substitution to *

(b) 3

$$\sqrt{3}\cos x - \sin x = A\cos(x+\alpha)$$

$$= 2(\cos x\cos\alpha - \sin x \sin\alpha)$$

$$A = \sqrt{(\sqrt{3})^2 + (-1)^2}$$

$$= 2$$

$$\therefore \sqrt{3}\cos x = 2\cos x\cos\alpha$$

$$\cos\alpha = \frac{\sqrt{3}}{2}$$

$$-\sin x = -2\sin x\sin\alpha$$

$$\sin\alpha = 1/2 \qquad \therefore \alpha = \pi/6$$

$$\therefore \sqrt{3}\cos x - \sin x = 2\cos(x + \pi/6)$$

$$2\cos(x+\pi/6) = -1$$

$$\cos(x+\pi/6) = -1/2$$

$$x + \pi/6 = \frac{2\pi}{3}, \frac{4\pi}{3}$$

$$\therefore x = \pi/2, \frac{7\pi}{6}$$

3	Correct
2	Correct A and α and progress towards sol^n
1	Correct A and α

Question 12 continues on page 8

Question 12 continued **Marks**

(c)

(i) 2

$r'(t) = (V\cos\theta)\underset{\sim}{i} + (V\sin\theta - gt)\underset{\sim}{j}$

Maximum height occurs when $\dot{y} = 0$

$\therefore V\sin\theta - gt = 0$

$t = \frac{V\sin\theta}{g}$

When $t = \frac{V\sin\theta}{g}$ $y = \frac{V\sin\theta}{g} \times V\sin\theta - \frac{1}{2}g\left(\frac{V\sin\theta}{g}\right)^2$

$= \frac{V^2\sin^2\theta}{g} - \frac{g}{2} \times \frac{V^2\sin^2\theta}{g^2}$

$= \frac{2V^2\sin^2\theta}{2g} - \frac{V^2\sin^2\theta}{2g}$

$= \frac{V^2\sin^2\theta}{2g}$ As req'd

2	Correct
1	Finding t value

Question 12 continues on page 9

Question 12 continued **Marks**

(c)

(ii) 2

When $y = 0$ $\quad Vt\sin\theta - \frac{1}{2}gt^2 = 0$

$$t\left(V\sin\theta - \frac{g}{2}t\right) = 0$$

$$\therefore t = 0 \quad \text{or} \quad \frac{g}{2}t = V\sin\theta$$

$$t = \frac{2V\sin\theta}{g}$$

When $t = \frac{2V\sin\theta}{g}$ $\quad x = V\cos\theta \times \frac{2V\sin\theta}{g}$

$$x = \frac{V^2}{g}\sin 2\theta$$

$\therefore$ The range is $\frac{V^2}{g}\sin 2\theta$

2	Correct
1	Find t

Question 12 continues on page 10

Question 12 continued

Marks

(d) 3

Let the statement be $4^n + 14$ is divisible by 6 for all $n \geq 1$

When $n = 1$

$4^1 + 14 = 18$ which is divisible by 6

Assume the statement is true for $n = k$

i.e $4^k + 14 = 6M$ $M \in \mathbb{Z}$, $k \geq 1$

RTP the statement is true for $n = k+1$

i.e $4^{k+1} + 14$ is divisible by 6

Now $4^{k+1} + 14$

$= 4 \times 4^k + 14$

From assumption, $4^k = 6M - 14$

Substituting

$\therefore = 4 \times (6M - 14) + 14$

$= 24M - 56 + 14$

$= 24M - 42$

$= 6(4M - 7)$

Since $M \in \mathbb{Z}$, $4M - 7 \in \mathbb{Z}$ $\therefore 4^{k+1} + 14$ is divisible by 6

Hence, since the statement is true for $n = 1$, k and $n = k+1$, the statement is true for $n = 1+1 = 2$ and so on by the principle of mathematical induction.

3	correct
2	Set up, conclusion and some progress
1	Set up and conclusion

Question 12 continues on page 11

Question 12 continued **Marks**

(e)

(i) 2

$P(0) = 0 \quad \therefore \quad d = 0$

$P(4) = 0 \quad \therefore \quad 64 + 16b + 4c = 0$

$16b + 4c = -64$

$4b + c = -16 \quad — (1)$

$P(-4) = 0 \quad \therefore \quad -64 + 16b - 4c = 0$

$16b - 4c = 64$

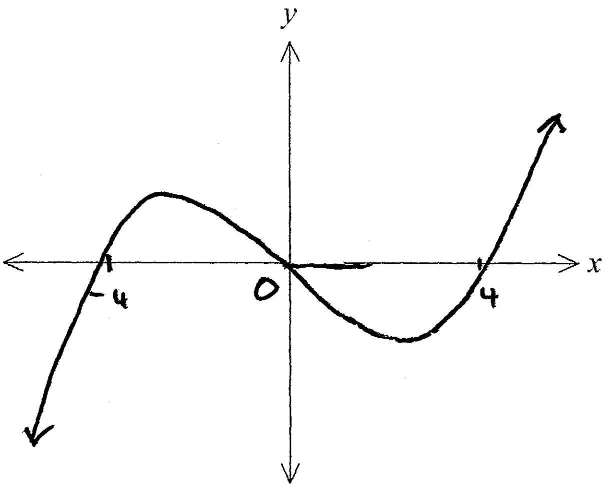
$4b - c = 16 \quad — (2)$

$(1) + (2) \quad 8b = 0 \quad$ From (1)

$\therefore b = 0 \quad c = -16 \quad \therefore P(x) = x^3 - 16x$

2	Correct
1	2 correct Values

(ii) 1



$P(x) = x(x^2 - 16)$

$= x(x-4)(x+4)$

1 r/w

(iii) 2

$\frac{x^2 - 16}{x} \leq 0$

This is the same as finding where $\frac{P(x)}{x^2} \leq 0$, since $x^2 \geq 0$, the solution to the inequality is $x \leq -4$ or $0 < x \leq 4$

2	Correct
1	Some progress

End of Question 12

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Question 13 (14 marks) **Marks**

(a) $\left(2x + \frac{5}{x^2}\right)^6 = \sum_{r=0}^{6} {}^6C_r (2x)^r (5x^{-2})^{6-r} = \sum_{r=0}^{6} {}^6C_r 2^r . 5^{6-r} x^{3r-12}$

2	correct
1	Working towards either x^3 or constant term

x^3 term $\Rightarrow {}^6C_5\, 2^5 . 5^1\, x^3$

Constant term $\Rightarrow {}^6C_4\, 2^4 . 5^2\, x^0$

$\therefore$ Term independent of x in $\left(2 - \frac{3}{x^3}\right)\left(2x + \frac{5}{x^2}\right)^6$

$= -3 \times 6 \times 2^5 \times 5 + 2 \times 15 \times 2^4 \times 5^2$

$= -2880 + 12000$

$= 9120$

(b)

(i) 2

$\overrightarrow{AB} = \overrightarrow{AM} + \overrightarrow{MB}$ $\quad$ $\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB}$

$= \underset{\sim}{a} - \underset{\sim}{b}$ $\quad$ $= 2\underset{\sim}{a} - (\underset{\sim}{a} - \underset{\sim}{b})$

$= \underset{\sim}{a} + \underset{\sim}{b}$

2	correct
1	Just one of the vectors $\overrightarrow{AB}$ or $\overrightarrow{BC}$

(ii) 2

Since $\triangle ABC$ is right-angled

$|\overrightarrow{AC}|^2 = |\overrightarrow{AB}|^2 + |\overrightarrow{BC}|^2$

$4|\underset{\sim}{a}|^2 = |\underset{\sim}{a} + \underset{\sim}{b}|^2 + |\underset{\sim}{a} - \underset{\sim}{b}|^2$

$4|\underset{\sim}{a}|^2 = |\underset{\sim}{a}|^2 + 2|\underset{\sim}{a}||\underset{\sim}{b}| + |\underset{\sim}{b}|^2 + |\underset{\sim}{a}|^2 - 2|\underset{\sim}{a}||\underset{\sim}{b}| + |\underset{\sim}{b}|^2$

2	correct
1	Correct expansion of Pythagoras' in vector form

$2|\underset{\sim}{a}|^2 = 2|\underset{\sim}{b}|^2$

$|\underset{\sim}{a}|^2 = |\underset{\sim}{b}|^2$ since $|\underset{\sim}{a}| > 0$ and $|\underset{\sim}{b}| > 0$

$|\underset{\sim}{a}| = |\underset{\sim}{b}|$ and since M is the midpoint of $\overrightarrow{AC}$ $\overrightarrow{AM} = \overrightarrow{MC} = \underset{\sim}{a}$

$\therefore$ A, B, C are equidistant from M

Question 13 continues on page 16

Question 13 continued **Marks**

(c)

(i) 2

$x = 1 - \sin t \quad y = \cos t \sin 2t$

When $x = 0$ When $y = 0$

2	Correct
1	Letting $x, y = 0$ and solving for t

$\sin t = 1 \qquad \cos t = 0$ or $\sin 2t = 0$

$t = \pi/2 \qquad t = \pi/2$ or $2t = 0, \pi, 2\pi$

$t = 0, \pi/2, \pi$

When $t = 0 \quad x = 1 \quad y = 0 \qquad \therefore x$ intercepts

$t = \pi/2 \quad x = 0 \quad y = 0 \qquad$ are 1 and 0.

$t = \pi \quad x = 1, \ y = 0 \qquad y$ intercept is 0

(ii) 1

$x = 1 - \sin t \qquad 0 \le t \le \pi$

Domain $\quad 0 \le x \le 1 \qquad$ 1 r/w

(iii) 2

2	Correct
1	Progress towards removing t

$y = \cos t \times 2 \sin t \cos t$

$= 2 \sin t \cos^2 t \qquad \sin t = 1 - x$

$= 2 \sin t (1 - \sin^2 t)$

$= 2(1-x)(1-(1-x)^2)$

$= 2(1-x)(1-(1-2x+x^2))$

$= 2(1-x)(2x - x^2)$

$y = 2x(1-x)(2-x), \ 0 \le x \le 1$

OR $y = 2x^3 - 6x^2 + 4x \ ; \ 0 \le x \le 1.$

Question 13 continues on page 17

Question 13 continued. **Marks**

(d)

(i) 1

$y = \ln(e+1-1)$ 1 r/w

$= 1$

(ii) 2

$\int_1^{e+1} \ln(x-1)\,dx$

$y = \ln(x-1)$

$x - 1 = e^y$

$x = e^y + 1$

$= (e+1)\times 1 - \int_0^1 (e^y+1)\,dy$

$= e+1 - \left[e^y + y\right]_0^1$

$= e+1-(e+1-e^0)$

$= 1 \text{ unit}^2$

2	Correct
1	Some progress by finding the difference of areas or using by parts

or

$u = \ln(x-1)$ $v' = 1$

$u' = \frac{1}{x-1}$ $v = x$

$A = \left[x\ln(x-1)\right]_2^{e+1} - \int_2^{e+1} \frac{x}{x-1}\,dx$

$= e+1-0-\int_2^{e+1}\left(\frac{x-1}{x-1}+\frac{1}{x-1}\right)dx$

End of Question 13

$= e+1-\left[x+\ln(x-1)\right]_2^{e+1}$

$= e+1-\left[e+1+1-(2+0)\right]$

$= 1 \text{ unit}^2$

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Question 14 (12 marks)

Marks

(a)

(i) 1

RHS : $\frac{1}{50A} + \frac{1}{50(50-A)}$ 1 r/w

$= \frac{50 - A + A}{50A(50-A)}$

$= \frac{50}{50A(50-A)}$

$= \frac{1}{A(50-A)}$

$=$ LHS

Question 14 continues on page 22

Question 14 **Marks**

Part (a) continued

(ii) $\frac{dA}{dt} = \frac{A}{25}(50 - A)$ **3**

$\int \frac{dA}{A(50-A)} = \int \frac{1}{25}\, dt$

$\frac{25}{50} \int \left(\frac{1}{A} + \frac{1}{50-A}\right) dA = t + C$

$\frac{1}{2}\left[\ln|A| - \ln|50 - A|\right] = t + C$

When $t = 0$ $A = 0.5$

$\therefore \frac{1}{2}\left\{\ln\left|\frac{0.5}{49.5}\right|\right\} = c$

$c = \frac{1}{2}\ln\frac{1}{99}$

$* \therefore \frac{1}{2}\ln\left|\frac{A}{50-A}\right| = t + \frac{1}{2}\ln\frac{1}{99}$

$2t = \left(\ln\left|\frac{A}{50-A}\right| - \ln\frac{1}{99}\right)$

$2t = \ln\left|\frac{99A}{50-A}\right|$

$** \quad e^{2t} = \frac{99A}{50-A}$

$e^{2t}(50 - A) = 99A$

$50e^{2t} - Ae^{2t} = 99A$

$A(99 + e^{2t}) = 50e^{2t}$

$A = \frac{50e^{2t}}{99 + e^{2t}} \quad \div \frac{e^{2t}}{e^{2t}}$

$A = \frac{50}{99e^{-2t} + 1}$ As req'd

1	subst from (i)
2	$e^{2t} = \frac{99A}{50-A}$ or equivalent
3	correct soln

Question 14 continues on page 23

Question 14

Part (a) continued

Marks

(iii) 1

As $t \to \infty$ $A \to \frac{50}{0+1}$ 1 r/w

$\to 50 \text{ cm}^2$

(b) 2

$n \to$ female $n \to$ male

P.i Day committee: 3 male, 2 male 1 female, 1 male 2 female, 3 female

$\therefore {}^nC_3 + {}^nC_2 \times {}^nC_1 + {}^nC_1 \times {}^nC_2 + {}^nC_3$

$= 2 \times {}^nC_3 + 2 \times {}^nC_2 \times {}^nC_1$

${}^nC_1 = n$

$\therefore = 2({}^nC_3 + n \times {}^nC_2)$

P.i Day committee: ${}^{2n}C_3$

$\therefore {}^{2n}C_3 = 2({}^nC_3 + n \times {}^nC_2)$

$\frac{1}{2}\, {}^{2n}C_3 = {}^nC_3 + n \times {}^nC_2$ As req'd

Lists outcomes
3M ${}^nC_3$
2M1F ${}^nC_2 \times {}^nC_1$
1M2F ${}^nC_1 \times {}^nC_2$
3M ${}^nC_3$

2 correct soln

Question 14 continues on page 24

Question 14 continued **Marks**

(c)

(i) 2

$\cos 4\theta = \cos(2\theta + 2\theta)$

$= \cos^2 2\theta - \sin^2 2\theta$

$= \cos^2 2\theta - (1 - \cos^2 2\theta)$

$= 2\cos^2 2\theta - 1$

$= 2(\cos^2\theta - \sin^2\theta)^2 - 1$

$= 2(\cos^4\theta - 2\cos^2\theta\sin^2\theta + \sin^4\theta) - 1$

$= 2(\cos^4\theta - 2\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2) - 1$

$= 2(\cos^4\theta - 2\cos^2\theta + 2\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta) - 1$

1 | applies correct identity

$= 2(4\cos^4\theta - 4\cos^2\theta + 1) - 1$

2 | correct soln

$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 2 - 1$

$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$ As req'd

(ii) 3

Let $x = \cos\theta$

$16\cos^4\theta - 16\cos^2\theta + 2 - \sqrt{2} = 0$

$8\cos^4\theta - 8\cos^2\theta + 1 = \frac{\sqrt{2}}{2}$

$* \therefore \cos 4\theta = \frac{\sqrt{2}}{2}$ $\therefore$ solving the eq^n is equivalent to finding the 4 distinct values of $\cos 4\theta = \frac{\sqrt{2}}{2}$

$4\theta = \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4}$

$\theta = \pi/16, \frac{7\pi}{16}, 9\pi/16, 15\pi/16$

$\cos\frac{9\pi}{16} = -\cos\frac{7\pi}{16}$ $\cos\frac{15\pi}{16} = -\cos\frac{\pi}{16}$

1 | substitute $\cos\theta$ and manipulates eqn

Roots are $\pm\cos\frac{\pi}{16}, \pm\cos\frac{7\pi}{16}$

Product of roots $\cos^2\frac{\pi}{16}\cos^2\frac{7\pi}{16} = \frac{2-\sqrt{2}}{16}$

2 | $\theta = \frac{\pi}{16}, \frac{7\pi}{16}, \frac{9\pi}{16}, \frac{15\pi}{6}$ must have 4 solns

As req'd.

3 | correctly reaches conclusion

END OF PAPER

If you use this space, clearly indicate which question you are answering.

14 c) i) quicker solution

$\cos 4\theta = 2\cos^2 2\theta - 1$

$= 2\left[2\cos^2\theta - 1\right]^2 - 1$

$= 2(4\cos^4\theta - 4\cos^2\theta - 1) - 1$

$= 8\cos^4\theta - 8\cos^2\theta - 2 + 1$

$= 8\cos^4\theta - 8\cos^2\theta - 1$

$=$ RHS as required